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On some group properties of structure equations of stellar systems

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Abstract. By using Lie group theory, symmetries of the system of equations describing the stellar system are investigated. The most general quasihomologous transformations are found. The symmetries resulting from them enforce a corresponding temporal evolution of the mass of the system.

1. Introduction

In the group analysis of differential equations, one exploits the algebraic structure induced in the set of solutions of a given equation by its symmetry group. By using this method one can construct new solutions from the known ones. The method of group analysis has successfully been used in hydrodynamics [1]. It has been applied in astrophysics to investigate symmetries of structure equations of Newtonian stars in the radiative equilibrium [2]. It has been noticed in [2] that basic physical relations such as the Hertzspring-Russell diagram or the Eddington mass-luminosity dependence are associated with the existence of certain invariants of the structure equations. Our intention is to develop the idea that invariants of the structure equations may provide some important physical relations.

In the present paper we apply the Lie group theory to equations of structure of a stellar system. We consider stellar systems which consists of stars with an equal mass. The flow of energy from the core to the halo is approximated by a gaseous model. Following Hachisu *et al* [3] we model the system as a conducting gaseous sphere. Effects of two-body encounters are modelled by a suitable choice of thermal conductivity.

Evolution of such systems has been extensively investigated. Hachisu *et al* [3] found that the collapse proceeded in a nearly self-similar way. The simplifications arising from this self-similarity were exploited by Lynden-Bell and Eggleton [4], who derived a dimensionless model for the late stages of core collapse. Inagaki and Lynden-Bell [5] constructed a self-similar postcollapse model that can be viewed as a continuation of their self-similar precollapse model of [4]. Goodman [6] constructed a self-similar solution for the asymptotic evolution of a system long after core collapse. Oscillations in the size of the core radius have been found numerically by Bettweiser and Sugimoto [7]. This result was confirmed by Goodman [8] who also presented a new regular self-similar model for postcollapse evolution. The problem of gravothermal oscillations provides an important stimulus for devising more realistic models of evolution. A more detailed review of these problems is given in [9].

Although the role that similarity solutions play in science is as general as the role of dimensional arguments, the demand of self-similarity is a very strong constraint imposed on the solution of differential equations. We therefore introduce the notion of quasihomology as the simplest extension of self-similarity. We obtain the operator of infinitesimal transformations leaving the structure equations invariant. Expressions for the temporal evolution of the mass of the system are given and homology theorems are stated allowing one to construct new families of solutions from the known ones. It is quite possible that the obtained results could be used, in a similar manner to that of Chandrasekhar [10] in the theory of stellar interiors.

The material is organised as follows. Section 2 gives necessary rudiments of the group method to analyse symmetries of differential equations. In § 3 we consider group properties of structure equations of a stellar system. Conclusions are presented in § 4.

2. Group method of investigating symmetries of differential equations

Let us consider a Lie group G of point transformations in the space (x, u), where $x=(x^1,\ldots,x^n)$ and $u=(u^1,\ldots,u^m)$, with the infinitesimal operator

$$X=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}\qquad i=1,\ldots,n;\ \alpha=1,\ldots,m. \tag{1}$$

Now, let us consider the space $(x, u, \underset{1}{u})$, where

$$\underset{1}{u}=\left\{u_i^\alpha=\frac{\partial u^\alpha}{\partial x^i};\ i=1,\ldots,n;\ \alpha=1,\ldots,m\right\}.$$

The action of the group G extends itself onto $\underset{1}{u}$, and we can describe it by introducing an extended group $\underset{1}{\mathrm{G}}$ acting in the space $(x, \underset{1}{u}, u)$. Its infinitesimal operator is

$$\underset{1}{X}=X\zeta_i^\alpha(x,u,\underset{1}{u})\frac{\partial}{\partial u_i^\alpha}. \tag{2}$$

The demand that formulae $u_i^\alpha=\partial u^\alpha/\partial x^i$ should be conserved under the action of group $\underset{1}{\mathrm{G}}$ determines functions ζ_i^α in the following way:

$$\zeta_i^\alpha=D_i(\eta^\alpha)-u_j^\alpha D_i(\xi^j) \tag{3}$$

where $D_i=\partial/\partial x^i+u_i^\alpha\,\partial/\partial u_\alpha$ is the operator of full differentiation with respect to x^i [1, 11]. Operator (2), such that (3) is fulfilled, is called an extension of X to the first derivatives. Analogously, we can extend the action of group G to the sth derivatives and construct respective extended operators $\underset{s}{X}$.

Given an sth-order partial differential equation

$$F(x,u(x),\ldots,\underset{s}{u}(x))=0 \tag{4}$$

the equation (4) defines a certain manifold M in the space $(x, u, \underset{1}{u},\ldots,\underset{s}{u})$. We say that equation (4) is invariant with respect to the group G if the manifold M is invariant with respect to the sth extension of G, i.e. $\underset{s}{\mathrm{G}}(M)=M$. In terms of infinitesimal operators it means that

$$\underset{s}{X}F|_{F=0}=0. \tag{5}$$

Condition (5) forms a system of linear homogeneous differential equations determining ξ and η, coordinates of the operator X. The point transformation generated by X is called quasihomologous if $\xi^i = \xi^i(x^i)$ and $\eta^\alpha = \eta^\alpha(u^\alpha)$ (no sum over i or α) and is a generalisation of the homologous one, for which $\xi^i(x^i)$ and $\eta^\alpha(u^\alpha)$ are linear functions. Hereafter in this paper we shall consider quasihomologous transformations.

3. Symmetries of structure equations of stellar systems

Stellar systems can be treated as gas spheres in which the energy is transferred by thermal conductivity, in this way modelling two-body encounters within the system [4].

The structure equations are the following:

$$\frac{\partial p}{\partial m} = -\frac{Gm}{4\pi r^4} \qquad \text{(hydrostatic equilibrium)} \tag{6a}$$

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi\rho r^2} \qquad \text{(mass continuity)} \tag{6b}$$

$$\frac{\partial T}{\partial m} = -\frac{1}{\rho K}\frac{l}{16\pi^2\rho r^4} \tag{6c}$$

$$\frac{\partial l}{\partial m} = -T\frac{Ds}{Dt} \tag{6d}$$

where m is the mass within the sphere of radius r, ρ the density, p the pressure, T the temperature, l the luminosity at the surface of the sphere of radius r, K the thermal conductivity coefficient, G the gravitational constant and s the entropy.

System (6) is valid outside the core because in (6d) there are no terms connected with sources of energy such as so-called hard binary systems playing an important role in the core.

It is convenient to rewrite equations (6) in the following parametrisation [4]:

$$T = \sigma^2 \qquad p = \rho\sigma^2 \qquad s = \ln(\sigma^3/\rho)$$

where σ is the one-dimensional velocity dispersion. The thermal conductivity coefficient K can be chosen as $K = AG/\sigma$ [4], where $A = 0.708 m_* \ln(0.4N)$, m_* is the mass of a single body and N is the total number of bodies in the system. It can be assumed that $A = \text{constant}$ [4].

By replacing the variable m in equations (6) by $x = m(r,t)/M(t)$, where M is the total mass of the system, the Euler derivative can be written in the form

$$\frac{D}{Dt} = \frac{\partial}{\partial t}\bigg|_x + \frac{\dot{M}}{M}(1-x)\frac{\partial}{\partial x}\bigg|_t .$$

Equations (6) now assume the form

$$\frac{\partial \rho\sigma^2}{\partial x} = -\frac{GM^2 x}{4\pi r^4} \tag{7a}$$

$$\frac{\partial r}{\partial x} = \frac{M}{4\pi\rho r^2} \tag{7b}$$

$$\frac{\partial \sigma}{\partial x} = -\frac{M}{AG}\frac{1}{32\pi^2\rho^2 r^4} \tag{7c}$$

$$\frac{\partial l}{\partial x} = -M\sigma^2\frac{\partial}{\partial t}\ln\left(\frac{\sigma^3}{\rho}\right) - \dot{M}\sigma^2(1-x)\frac{\partial}{\partial x}\ln\left(\frac{\sigma^3}{\rho}\right). \tag{7d}$$

We now look for the symmetry transformations of equations (7) generated by the operator

$$X=\xi^1(x)\frac{\partial}{\partial x}+\xi^2(t)\frac{\partial}{\partial t}+\eta^1(\rho)\frac{\partial}{\partial\rho}+\eta^2(r)\frac{\partial}{\partial r}+\eta^3(\sigma)\frac{\partial}{\partial\sigma}+\eta^4(l)\frac{\partial}{\partial l}. \tag{8}$$

After having determined the operator $\underset{1}{X}$, according to (2) and (3), we obtain the admissible equations (5) in the form

$$\frac{G}{4\pi}\frac{M^2x}{\sigma^2r^4}\left(\frac{\mathrm{d}\xi^1}{\mathrm{d}x}-\frac{\mathrm{d}\eta^1}{\mathrm{d}\rho}+2\frac{\dot M}{M}\xi^2+\frac{\xi^1}{x}-\frac{2\eta^3}{\sigma}-\frac{4\eta^2}{r}\right)$$
$$-\frac{Ml}{16\pi^2AGr^4\sigma\rho}\left(-\frac{\mathrm{d}\eta^1}{\mathrm{d}\rho}+\frac{\mathrm{d}\xi^1}{\mathrm{d}x}+\frac{\dot M}{M}\xi^2-\frac{\eta^1}{\rho}-\frac{\eta^3}{\sigma}+\eta^4-\frac{4\eta^2}{r}\right)=0 \tag{9a}$$

$$\frac{\mathrm{d}\eta^2}{\mathrm{d}r}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}-\frac{\dot M}{M}\xi^2+\frac{\eta^1}{\rho}+\frac{2\eta^2}{r}=0 \tag{9b}$$

$$\frac{\mathrm{d}\xi^1}{\mathrm{d}x}-\frac{\mathrm{d}\eta^3}{\mathrm{d}\sigma}+\frac{\dot M}{M}\xi^2+\frac{\eta^4}{l}-\frac{2\eta^1}{\rho}-\frac{4\eta^2}{r}=0 \tag{9c}$$

$$\frac{\partial l}{\partial x}\left(\frac{\mathrm{d}\eta^4}{\mathrm{d}l}-\frac{\mathrm{d}\xi^1}{\mathrm{d}x}\right)+3M\sigma\frac{\partial\sigma}{\partial t}\left(\frac{\mathrm{d}\eta^3}{\mathrm{d}\sigma}-\frac{\mathrm{d}\xi^2}{\mathrm{d}t}\right)-\frac{M\sigma^2}{\rho}\frac{\partial\rho}{\partial t}\left(\frac{\mathrm{d}\eta^1}{\mathrm{d}\rho}-\frac{\mathrm{d}\xi^2}{\mathrm{d}t}\right)$$
$$+3\frac{\partial\sigma}{\partial t}(\sigma\dot M\xi^2+M\eta^3)-\frac{\partial\rho}{\partial t}\left(-\frac{M\sigma^2}{\rho^2}\eta^1+\frac{\dot M\sigma^2}{\rho}\xi^2+\frac{2M\sigma}{\rho}\eta^3\right)$$
$$-\frac{5}{32AG\pi^2}\left(-\frac{M\dot M\sigma l}{\rho^2r^4}\xi^1+(M\dot M)^{\cdot}\xi^2\frac{\sigma l(1-x)}{\rho^2r^4}-\frac{2M\dot M\sigma l(1-x)}{\rho^3r^4}\eta^1\right.$$
$$\left.-\frac{4M\dot M\sigma l(1-x)}{\rho^2r^5}\eta^2+\frac{M\dot Ml(1-x)}{\rho^2r^4}\eta^3+\frac{M\dot M\sigma(1-x)}{\rho^2r^4}\eta^4\right)$$
$$+\frac{G}{4\pi}\left(\frac{M^2\dot M}{\rho r^4}(1-2x)\xi^1+\frac{x(1-x)}{\rho r^4}(M^2\dot M)^{\cdot}\xi^2\right.$$
$$\left.-\frac{4M^2\dot Mx(1-x)}{\rho r^5}\eta^2-\frac{M^2\dot Mx(1-x)}{\rho^2r^4}\eta^1\right)=0. \tag{9d}$$

From (9b) we can see that $\eta^1=\alpha_1\rho$, $\eta^2=\alpha_2 r$ and $\xi^1=a_1x+b_1$, where α_1, α_2, a_1, b_1 are constants, and that

$$\frac{\dot M}{M}\xi^2=\varphi_2=\text{constant}. \tag{10}$$

Formula (10) implies that the temporal evolution of the total mass of the system depends on the coordinate ξ^2 of the operator X:

$$M(t)=\exp\left(\varphi_2\int\frac{\mathrm{d}t}{\xi^2}\right).$$

In the particular case of homologous transformations, when $\xi^2 = a_2 t$ we obtain $M(t) = M_0 t^{\varphi_2/a_2}$ which corresponds to the dependence given by Goodman [6].

From (9*c*) we see that $\eta^4 = \alpha_4 l$, $\eta^3 = \alpha_3\sigma + \beta_3$, where $\alpha_3, \alpha_4, \beta_3$ are constants.

From (9*d*) one obtains:

$$\dot{\xi}^2 = \varphi_2 + \frac{\alpha_4}{M(t)} \tag{11}$$

$$\left(3\frac{\partial \ln \sigma}{\partial t} - \frac{\partial \ln \rho}{\partial t}\right)(a_1 - \alpha_4 + 2\alpha_3 - \dot{\xi}^2 + \varphi_2) = 0. \tag{12}$$

Formula (11) gives us a differential equation for the coordinate ξ^2. Equation (12) implies that $3\partial \ln \sigma/\partial t = \partial \ln \rho/\partial t$ since equating the second factor to zero leads to a contradiction. This means that $\partial s/\partial t = 0$, i.e.

$$\frac{Ds}{Dt} = \frac{\dot{M}}{M}(1-x)\frac{\partial s}{\partial x}.$$

By substituting the components of X into (9*a*)-(9*d*) we obtain that

$$\alpha_2 = -\tfrac{5}{2}\alpha_1 \qquad \alpha_3 = -2\alpha_1 \qquad \alpha_4 = -\tfrac{7}{2}\alpha_1 \qquad \varphi_2 = -\tfrac{13}{2}\alpha_1. \tag{13}$$

All remaining constants are equal to zero.

In order to determine ξ^2 we have to solve the system of differential equations (10) and (11).

Expressing ξ^2 by M, we obtain:

$$\frac{d\xi^2}{dM} = \left(\frac{1}{M} + \frac{\beta}{M^2}\right)\xi^2$$

where $\beta = \alpha_4/\varphi^2 = \frac{7}{13}$. This equation can be integrated by separating the variables to obtain

$$\xi^2(t) = \xi_0 M(t)\exp(-\beta/M(t)) \tag{14}$$

where ξ_0 is an integration constant.

By substituting (14) into (10) we obtain

$$\frac{dM}{dt} = \frac{\varphi_2}{\xi_0}\exp(\beta/M). \tag{15}$$

If $\varphi_2/\xi_0 < 0$, the loss of mass from the centre decreases the binding energy of the system.

To integrate equation (15) one separates the variables and obtains

$$M\exp(-\beta/M) - \beta\ln M + \sum_{n=1}^{\infty}\frac{(-\beta)^{n+1}}{nn!}M^{-n} = \frac{\varphi_2}{\xi_0}t + t_0 \tag{16}$$

where t_0 is an integration constant.

Formula (16) gives us an implicit form of $M(t)$, i.e. it describes temporal evolution of the total mass of the system.

Having the operator X determined:

$$X = \xi_0\exp(-\beta/M(t))M(t)\frac{\partial}{\partial t} + \alpha_1\rho\frac{\partial}{\partial\rho} - \tfrac{5}{2}\alpha_1 r\frac{\partial}{\partial r} - 2\alpha_1\sigma^2\frac{\partial}{\partial\sigma} - \tfrac{7}{2}\alpha_1 l\frac{\partial}{\partial l} \tag{17}$$

we obtain finite transformations of variables t, ρ, r, σ and l which leave the form of the structure equations (7) invariant. These transformations are as follows:

$$\bar{t} = \exp\left(\xi^2(t)\frac{\partial}{\partial t}\right)t = \sum_{n=0}^{\infty}\frac{l}{n!}\left(\xi^2\frac{\partial}{\partial t}\right)^{(n)} t$$
$$\bar{\rho} = \rho \exp \alpha_1 \qquad \bar{r} = r\exp(-\tfrac{5}{2}\alpha_1) \tag{18}$$
$$\bar{\sigma} = \sigma\exp(-2\alpha_1) \qquad \bar{l} = l\exp(-\tfrac{7}{2}\alpha_1).$$

Operator (17) possesses four independent invariants which are solutions of the following system:

$$-\frac{2}{13}\frac{\mathrm{d}t}{(\ln M)^{\cdot}} = \frac{\mathrm{d}\rho}{\rho} = -\frac{2}{5}\frac{\mathrm{d}r}{r} = -\frac{1}{2}\frac{\mathrm{d}\sigma}{\sigma} = -\frac{2}{7}\frac{\mathrm{d}l}{l}. \tag{19}$$

In the first term of (19) we have taken into account the formula (10). These invariants can be chosen in the following way:

$$\mathcal{I}_1 = \rho M^{2/13} \qquad \mathcal{I}_2 = rM^{-5/13} \qquad \mathcal{I}_3 = \sigma M^{-4/13} \qquad \mathcal{I}_4 = lM^{-7/13}. \tag{20}$$

By using these invariants and the finite transformations (18) we can now formulate an homology theorem describing the construction of new families of solutions.

Theorem. If $\rho(t)$, $r(t)$, $\sigma(t)$, $l(t)$ are solutions of system (7), then also

$$\rho(\bar{t})M(\bar{t})^{2/13}\exp\alpha_1 \qquad r(\bar{t})M(\bar{t})^{-5/13}\exp(-\tfrac{5}{2}\alpha_1)$$
$$\sigma(\bar{t})M(\bar{t})^{-4/13}\exp(-2\alpha_1) \qquad l(\bar{t})M(\bar{t})^{-7/13}\exp(-\tfrac{7}{2}\alpha_1)$$

are solutions of the same system (α_1 is the group parameter). This theorem is analogous to homology theorems stated by Chandrasekhar [10] which play an important role in the theory of stellar interiors.

4. Conclusion

We have characterised, by computing an infinitesimal operator, the structure of the group admissible by a system of equations describing a stellar system modelled as a gas sphere. We have also shown that, in the most general case, the equations admit a one-parameter group of quasihomologous transformations. In the particular case of homologous transformations the results of Goodman [6] are recovered. Quasihomologous symmetries enforce an appropriate evolution of the mass of the system described by formula (16). By using this formula together with invariants (20) one can obtain the corresponding evolution of other physical parameters such as the density, the radius and the luminosity. Invariants given by (20) are basis invariants, i.e. any combination of them remains an invariant.

We have finally formulated a homology theorem describing the construction of new families of solutions from the known ones.

In the context of gravothermal oscillations, a more general form of the symmetry operator X should be considered, but this requires further investigation.

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